Bundling Strategies when Products are Vertically Differentiated and Capacities are Limited

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Abstract

We consider a seller who owns two capacity constrained resources and markets two products (components) corresponding to these resources as well as a bundle comprising the two components. In an environment where all customers agree that one of the two components is of higher quality than the other and that the bundle is of the highest quality, we derive the seller's optimal bundling strategy. We demonstrate that the optimal solution depends on the absolute and relative availabilities of the two resources as well as upon the extent of subadditivity of the quality of the products. The possible strategies that can arise as equilibrium behavior include a pure components strategy, a partial or a full spectrum mixed bundling strategy, and a pure bundling strategy, where the latter strategy is optimal when capacities are unconstrained. These conclusions are contrary to findings in the prior literature on bundling which demonstrated the unambiguous dominance of the full spectrum mixed bundling strategy. Our work expands the frontier of bundling to an environment with vertically differentiated components and limited resources. We also explore how the bundling strategies change as we introduce an element of horizontal differentiation wherein different types of customers value the available components differently.

Key words: vertical differentiation, limited resources, bundle pricing, mixed bundling, revenue management.

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1. Introduction

Simon Property Group (SPG),[1] the largest shopping-mall owner in the US, leases mall space to retailers. Its portfolio of properties includes *premium* malls such as the Copley Place in the exclusive BackBay area of Boston, MA as well as *normal* malls in the same market such as the Arsenal Mall just a few miles away. Premium malls are situated in more attractive and more accessible locations; they also attract shoppers with higher disposable incomes compared to normal malls. As a consequence, mall retailers consider premium-mall space to be more valuable than normal-mall space. However, a retailer's ability to generate higher revenues at premium malls varies greatly depending on the products it sells. For instance, Johnston and Murphy (J&M), and Aldo both sell men's shoes and accessories at both types of malls. However, J&M, appealing to the well-heeled professional, derives greater value from a premium mall store than does Aldo. Similarly, premium-mall locations generate greater value for Tiffany and Co. and Williams-Sonoma as compared to Zales Jewelers and the Kitchen Collection. Given this variability in the retailer valuations, SPG can more successfully extract rents from retailers by designing a product line that facilitates enhanced segmentation of the population of retailers. Specifically, in addition to offering separate store leases in each mall type (premium or normal), it can also offer at a discounted price a bundle (Stigler, 1963) consisting of two stores, one from each type of mall.[2]

With bundling as an option, a seller (e.g., the mall owner) can choose between a: (i) *pure components* strategy, that is, offer the products (components) only as separate items; (ii) *pure bundling* strategy, that is, offer only the package of the two components; and (iii) *mixed bundling* strategy, that is, offer both the bundle and the components. Mixed bundling offers the opportunity to more precisely segment the market, and previous literature has shown (Schmalensee, 1984) that this strategy (weakly) dominates the

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[1] http://www.simon.com/about_simon/index.aspx
[2] As evidence of bundling in this sector, consider the February 17, 2010 article about SPG's bid to acquire General Growth in which the *Wall Street Journal* reported "[SPG's] size would mean that retail chains such as Gap Inc. and AnnTaylor Corp. would have only one landlord to deal with when negotiating leases and opening stores at many high-end malls. A mall owner with hundreds of properties can pressure retailers into opening stores in struggling locations as a condition of getting space at its choicest sites."

other two strategies. However, in practice, we see marketers using all three strategies. This paper examines this heterogeneity in the use of bundling by deriving optimal revenue-maximizing strategies for sellers such as SPG, who offer vertically differentiated products in a limited capacity environment.

Mall retailers consider a premium mall store to be more attractive than a normal mall store, and a bundle of both stores to be more attractive than the premium product by itself. Thus, given suitably low prices, all buyers prefer the normal product to no product, the premium product to the normal product, and the bundle to the premium product. This vertical differentiation exists in many other markets as well. In television advertising, advertisers consider prime-time to be more valuable than non-prime time since it traditionally attracts a higher number of viewers. As another example, IDS (of the Li and Fung group) operates pharmaceutical manufacturing plants in both Malaysia and Thailand.[3] Companies buy manufacturing capacity from IDS, and value capacity in Malaysia more highly than in Thailand due to better overall infrastructure[4] and better corporate governance (McGee, 2008). Obviously, in these situations, the bundle provides the highest value. The extant bundling literature has not studied this type of preference ordering, arising due to differing *quality ratings*, among the products.

The resource availability in settings such as shopping mall and contract manufacturing is limited. Likewise, in television advertising published data suggests that competitive pressures implicitly limit the commercial time networks use: The non-programming minutes per hour are 16:32, 16:36, 16:46, and 16:57 for Fox, CBS, NBC, and ABC respectively.[5] Based on these observations, we assume that the resource availabilities are limited, and investigate how the seller's decisions change with capacity. None of the previous bundling literature has modeled either vertical differentiation or resource availabilities.

The relationship among the quality ratings of the products is another important factor that distinguishes our work from previous bundling literature. For example, the number of unique shoppers

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[3] http://www.idsmanufacturing.com/
[4] http://siteresources.worldbank.org/INTEAPINFRASTRUCT/Resources/CH1.pdf. World Bank report. 2004.
[5] http://www.kued.org/misc/pdfs/outreach/readyToLearn/tvclutter.pdf. Report by American Association of Advertising Agencies and the Association of National Advertisers. 1999.

visiting Gap at a bundle of premium and normal mall locations in a particular trade area is likely to be lower than the sum of unique shoppers at each of the two stores if it were to open one without the other. In other words, the quality ratings have a subadditive relationship in the shopping mall application. As we argue later, other situations may have a superadditive quality relationship; we study how the nature of this relationship further affects the seller's optimal bundling strategy.

Certain characteristics of our problem (e.g., perishable resources, fixed capacities) suggest an overlap with the revenue management literature (see the comprehensive book by Talluri and van Ryzin (2004) and the survey by Bitran and Caldentey (2003)). However, our study differs significantly from traditional revenue management work in terms of the contextually-specified relationship between the quality ratings of the products, the basis for market segmentation, and the research focus. For example, previous revenue management literature focused primarily on travel related industries (airline, lodging, cruise, or car rental) where horizontal, not vertical, differentiation is prevalent. Even in other industries such as television advertising, the revenue management research focus is on optimizing advance sales (Bollapragada and Mallik, 2008) or capacity allocation between advance and spot markets (Araman and Popescu, 2010), rather than on modeling the vertically differentiated market. Furthermore, in this literature, customers are segmented a priori according to demographical or observed/forecasted behavioral traits (e.g. business/leisure travelers), and companies implement "fences" (e.g., a Saturday night stay a student id requirement) in order to prevent spillage between segments. Our approach, on the other hand, uses a self-selection mechanism for (second degree) price discrimination.

Our analysis combines vertically differentiated products and limited resource availability to obtain several novel and insightful results. First, when the quality ratings are subadditive, we show that as observed in practice, a full range of segmentation strategies can be optimal. This is contrary to past bundling literature which shows that the mixed bundling strategy (weakly) dominates the pure components and the pure bundling strategies. Interestingly, the tightness *and* the relative tightness of the available resources play a pivotal role in determining the optimal strategy. With superadditive quality ratings, the optimal set of strategies still depends on the resource availability but no longer spans full

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market segmentation or pure components. When capacities are unconstrained the optimal strategy is always pure bundling. Thus, the clean, unambiguous structure of the optimality of mixed bundling breaks down when the products are vertically differentiated and resources are limited. Second, whereas the earlier literature tied the benefit from mixed bundling primarily to the extent of heterogeneity in consumer preferences (Schmalensee, 1984) we demonstrate that the *skewness* of the distribution of preferences is also important. Third, we underscore how the degree of the subadditivity in the ratings affects the optimal strategy, showing that increased subadditivity increases the propensity to use the mixed bundling and the pure component strategies—this result is antithetical to our a priori intuition which suggests that increased subadditivity would lead to enhanced use of only the pure components strategy. Finally, to model the situation where quality ratings depend on the market segment, we develop and analyze a model with added horizontal differentiation. We demonstrate that increased horizontal product differentiation has an effect that is similar to reduced subadditivity of the ratings of the vertically differentiated components.

This paper is organized as follows. Section 2 discusses our modeling assumptions and develops a nonlinear pricing model. In Section 3, we analyze the optimal solution properties assuming a uniform distribution of the buyer's efficiency. Section 4 studies the impact of horizontal differentiation. Section 5 shows that our conclusions are quite robust and hold for other distributions as well. Section 6 concludes the paper by identifying some future research directions.

2. The Model

A monopolist shopping mall owner or a monopolist television broadcasting network,[6] both of whom we refer to generically as the *seller*, considers offering for sale her[7] available *resources*, which are

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[6] If SPG's acquisition of General Growth is successful, it would own more than a third of all malls and more than half of all premium malls (*The Wall Street Journal*, February 17, 2010, Simon Offers $10 Billion for General Growth). Coupled with high entry barriers in the shopping mall industry, this fact suggests that SPG is a virtual monopoly at least in some regions. Similarly, some television broadcasting market segments (channels), such as sports (ESPN), music (MTV), comedy (CC), and cooking (FOOD), exhibit a virtual monopolistic environment.
[7] Where necessary, we use feminine gender for the seller and masculine gender for the buyer.

of two types: *normal* and *premium*. The availability of both resources is fixed, with q_N (q_P) denoting the amount of normal (premium) resources available. The seller's objective is to maximize her total revenue. As in Bakos and Brynjolfsson (1999), we assume that the variable costs of both resources are zero.[8] The seller considers three products: the *normal* and the *premium products* consisting of one unit of the corresponding resources, and a *bundle* consisting of one unit of each of the two resources. These items can be advertising time during nonprime time (normal) and prime time (premium) segments, or retail stores in normal and premium malls.

The market consists of *buyers* interested in purchasing these three products. In line with the bundling literature (Adams and Yellen, 1976; Schmalensee, 1984), we assume that the marginal utility of a second unit of capacity of a given type is zero for all buyers.[9] Buyers have a strict ordering of their preferences: They prefer purchasing (i) a bundle to a premium product, (ii) a premium product to a normal product, and (iii) a normal product to refraining from purchasing altogether. We designate the quality ratings of the normal, premium, and the bundle options by α, β, and γ, respectively, where, $0 < \alpha < \beta < \gamma$.

Let $\sigma = (\gamma - \beta) / \alpha$ denote the *quality relationship* parameter. We consider both $\sigma < 1$ and $\sigma \geq 1$, representing respectively a subadditive and a superadditive relationship in the quality ratings. We say that ratings subadditivity increases (decreases) when σ decreases (increases). The premium and the normal shopping mall would attract shoppers from the same trade area and so the quality ratings exhibit a subadditive relationship. Similarly, Goettler (1999), and Brown and Cavazos (2003) observe empirically a subadditive relationship in an advertising setting where multiple showings of a television commercial increase the number of unique viewers less than proportionally. Jones' (1997) empirical analysis also

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[8] When the components are independently valued and resource availability is unlimited, unit costs have been considered earlier in the literature by McAfee et al. (1989), among others. Their results show that mixed bundling is the (weakly) dominant strategy, in contrast to the full range of optimal strategies that we obtain. We conjecture that for the situation we are modeling, unit component costs do not affect the general thrust of our findings.
[9] Keeping this assumption invariant while introducing vertical differentiation and limited resource capacities allows us to isolate the effect of these two aspects on the seller's marketing strategy, and contrast our findings with those in the existing literature. For a relaxation of this assumption, see Hitt and Chen (2005) and Wu et al. (2008).

indicates a subadditive relationship between product sales and the number of advertisements. On the

other hand, other researchers (McDonald, 1979; Naples, 1979) have argued that repeated exposures to the

same commercial may be necessary for modifying consumption habits. Hence superadditivity may be

more appropriate in this case. As well, quality ratings may reflect a superadditive relationship when there

are multiple decision makers who have different viewing preferences. For example, Mattel might

advertise during non-prime time to target children and during prime time to target the parent. To sell a

big ticket item such as an automobile or a large kitchen appliance, both spouses (who may have different

viewing habits) may need to be targeted, and so a company like Maytag may see advertisements during

prime time and non-prime time as complementing each other. Similarly, lower catastrophic risk

associated with a dual sourcing strategy by IDS's clients might argue for superadditivity.

Buyers differ in their willingness to pay for the three product variants. For example, a retailer

such as Abercrombie and Fitch may be more willing to lease a store in a premium mall than Dress Barn.

While both retailers value the larger volume and higher disposable income of potential customers at a

premium mall, Abercrombie and Fitch benefits to a greater extent since premium mall customers tend

typically to be more brand conscious. We designate by the parameter t the intrinsic efficiency of a buyer

to generate value from his customers, and assume that this efficiency is continuously distributed on the

unit interval according to some (differentiable) probability density function $f(t)$ and cumulative

distribution function $F(t)$. The willingness to pay of a buyer with efficiency t for product variant i is thus

equal to $t\,r_i$, where r_i is the quality rating of the ith product variant. These variants are the seller's normal

(N), premium (P), and bundle (B) options.

Given this distribution of intrinsic efficiency and the willingness to pay function, the seller's

optimal strategy induces the buyers to self-select into at most four segments as described in Figure 1, with

the thresholds T^*, T^{**}, and T^{***} demarcating the different market segments.[10] With this strategy, buyers

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[10] The willingness to pay function satisfies the "single crossing property" and therefore facilitates segmentation and
guarantees the uniqueness, as well as the monotonicity ($0 \leq T^{***} \leq T^{**} \leq T^* \leq 1$) of the thresholds.

in the highest range of intrinsic efficiencies (interval [T^*, 1]) choose to purchase the bundle. Those in the

second highest range (interval [T^{**}, T^*)) choose the premium product, those in the third highest range

(interval [T^{***}, T^{**})) choose the normal product, and those in the lowest range refrain from purchasing

altogether. An interval of zero length implies that the seller does not find it optimal to offer the

corresponding product. The values of the threshold parameters T^*, T^{**}, and T^{***} are determined to

guarantee that the buyer located at a given threshold level is indifferent between the products

corresponding to the two adjacent intervals separated by this threshold parameter.

PLEASE INSERT FIGURE 1 ABOUT HERE

To set up the model we define the selling prices for the bundle, premium, and normal products by

p_B, p_P and p_N, respectively. The seller's revenue optimization model with mixed bundling, *ROMB*, is:

$$[\text{ROMB}] \quad \max_{p_B, p_P, p_N \geq 0} \pi = p_B[1 - F(T^*)] + p_P[F(T^*) - F(T^{**})] + p_N[F(T^{**}) - F(T^{***})] \quad (1)$$

subject to:
$$p_B \leq p_P + p_N, \quad (2)$$

$$1 - F(T^{**}) \leq q_P, \text{ and} \quad (3)$$

$$1 - F(T^*) + F(T^{**}) - F(T^{***}) \leq q_N. \quad (4)$$

The seller's revenue from a market segment equals its size multiplied by the price of the

corresponding product; the total revenue, π, in (1) is the sum of the revenues from the three market

segments. Constraint (2), the "price-arbitrage" constraint, prevents arbitrage opportunities for a buyer to

compose a bundle by buying the premium and the normal products separately.[11] Constraints (3) and (4)

are capacity constraints for the premium and normal resources.

Buyers self-select their purchases (or decide against purchasing any product) based on their

willingness to pay and the product prices. (See Moorthy (1984) for an analysis of self-selection based

market segmentation.) We refer to the difference between a buyer's willingness to pay and the price of

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[11] Unless a systematic secondary market exists, an intermediary may not be able to purchase the components and sell the bundle at a profit. As a result, constraint (2) will not be necessary if secondary markets can be prevented and buyers cannot "assemble" the bundle from the components on their own.

the product he purchases as the *net benefit* that the buyer derives from the purchase. A buyer will

purchase a product only if his net benefit is nonnegative. Moreover, a buyer will be indifferent, say,

between buying the premium product and buying a bundle if he extracts the same net benefit from either

purchase. The following relationships (5) between the net benefits are invariant boundary conditions,

regardless of the efficiency distribution $f(t)$:

$$\gamma T^* - p_B = \beta T^* - p_P, \quad \beta T^{**} - p_P = \alpha T^{**} - p_N, \quad \text{and } \alpha T^{***} - p_N = 0. \quad (5)$$

These conditions imply $T^* = (p_B - p_P)/(\gamma - \beta)$, $T^{**} = (p_P - p_N)/(\beta - \alpha)$, and $T^{***} = p_N/\alpha$, and non-

negativity of the thresholds implies $0 \leq p_N \leq p_P \leq p_B$. Moreover, the net benefit for customers in each of

the three categories is nonnegative.

Before we analyze the situations that arise when at least one of the capacity constraints is binding,

Proposition 1 considers the case when neither capacity constraint is binding. Let $h(t) \equiv f(t) / [1 - F(t)]$ be

the *hazard* rate function. The Appendix gives the technicalities of this and all subsequent results.

Proposition 1. *If the premium and normal resource availability is sufficiently high, $th(t)$ is a*

monotonically increasing function of t on the domain [0, 1], *and $h(1) > 1$, the optimal strategy for the*

seller is pure bundling. The corresponding optimal threshold is the fixed point of the reciprocal of the

hazard rate function of the distribution of buyers, that is, $T^* = [1 - F(T^*)] / f(T^*)$.

The following corollary uses Markov's inequality to establish an upper bound on the optimal

revenue when the resource availability is unconstrained.

Corollary 2. *An upper bound on the seller's total revenue π is $\gamma \mathbf{E}[T]$, where $\mathbf{E}[T]$ is the expected value*

of the efficiency, t. The revenue collected under the pure bundling strategy is $\gamma[1 - F(T^*)] / h(T^*)$.

Note that the monotonicity condition on $th(t)$ and the condition $h(1) > 1$ are satisfied by a large

number of commonly used distributions such as the Beta (with both shape parameters at least 1), and the

Uniform and Normal truncated between 0 and 1. See the remark following the proof of Proposition 1 for

a more complete list. Thus, the result in Proposition 1 is quite general. Second, this result seems to

contradict previous bundling literature (for example, McAfee, McMillan and Whinston, 1989;

Schmalensee, 1984) which demonstrates that the mixed bundling strategy weakly dominates both the pure bundling and pure components strategies. This apparent contradiction can be explained by the fact that markets are vertically differentiated in our context. In contrast, previous research does not assume such an ordering. Since every buyer finds the bundle to be the most desirable option, the seller offers only the bundle when the available premium and normal resources are unconstrained. As mentioned previously, the unconstrained case is unlikely to arise in the context of our motivating examples. Broadcasters are constrained by the advertising time resource and mall operators are similarly capacity constrained.

In the next section we derive the analytical solution of the constrained optimization problem under the simplifying assumption that the efficiency parameter of buyers is uniformly distributed. In Section 5, we extend the results numerically using a Beta distribution.

3. Revenue Maximizing Strategies when Capacity is Binding

Clearly, the capacity constraints in the *ROMB* model play a significant role in determining the seller's optimal strategy. Particularly, the relative scarcity of the two resources, premium and normal, is the main driver of the analysis. For the shopping mall and the television advertising settings, the premium resource availability constraint (3) is more likely to be binding than the normal resource availability constraint (4) due to infrastructural and competitive reasons, respectively. This is not necessarily true for applications such as the IDS one, where the normal resource availability constraint could be binding if better incentives were provided by the Malaysian authorities for building capacity. In this section, we identify the impact of the capacity constraints on the seller's optimal strategy when the distribution of the efficiencies is uniform. We find that the following strategies can arise as the optimal solution of *ROMB*: the bundle is not offered, that is, the pure components strategy, *PC*; only the bundle is offered, that is, the pure bundling strategy, *PB*; the bundle, as well as each separate product is offered, that is, the full spectrum mixed bundling strategy, *MBPN*; the bundle and the premium product are offered, that is, the partial spectrum mixed bundling strategy, *MBP*; the bundle and the normal product are offered, that is, the partial spectrum mixed bundling strategy, *MBN*.

In our derivations, we will demonstrate that the optimal strategy critically depends upon the

relative availability of q_P and q_N. To capture this, we define the *capacity mismatch* (i. e, resource

mismatch) parameter $\rho = (1 - 2q_N) / (1 - 2q_P)$. (This non-intuitive form of the capacity mismatch

parameter will become apparent in the sequel.) When ρ exceeds one, q_N is scarce relative to q_P and the

opposite is true when ρ is less than one. The parameter ρ equals 1 when both capacities are equal. We

will show, for instance, that the *MBN* (*MBP*) strategy is optimal when ρ is sufficiently less (greater) than

1. The *MBPN* strategy is optimal when the relationship among the quality ratings is subadditive and ρ is

close to one, and q_N and q_P are both sufficiently large. We will also show that the characterization of the

solution when the partial spectrum mixed bundling strategies (*MBP* or *MBN*) are optimal is further

contingent upon the overall availability of the more abundant resource. Specifically, even though the

strategy itself, say *MBP*, remains the same, the solution characteristics (e.g., the product prices) depend

on whether q_P is less than or greater than a threshold value. To distinguish between these two cases, we

designate by MBP^+ and MBP^- the partial spectrum mixed bundling strategies when q_P is greater than and

when q_P is less than the threshold value, respectively. The threshold value depends on the efficiency

distribution and, for example, equals 1/2 for the uniform distribution. We define the subcategories MBN^+

and MBN^- of *MBN* in a similar manner depending on q_N.

Figure 2a depicts the regions corresponding to the various strategies when the relationship

between the quality ratings is subadditive, and Figure 2b depicts these regions when superadditivity

characterizes the ratings relationship. For the uniform distribution, the unconstrained solution that we

described in Section 2 arises when q_P and q_N are both at least a half. In this case, at the optimal solution,

the seller never sells more than an aggregate quantity of one, split equally between the premium and

normal product variants. To depict the constrained solution, therefore, in Figure 2, we restrict attention

only to the case when $q_P + q_N \leq 1$. The unconstrained solution in the figure is designated by the point,

PB, where $q_P = q_N = 1/2$. We will explain in detail the boundaries for the regions in Figure 2.

PLEASE INSERT FIGURE 2 ABOUT HERE

The model *ROMB* specializes to *ROMB_U* for the uniform distribution case.

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[ROMB_U] $\quad \max \pi = p_B\left(1 - \dfrac{p_B - p_P}{\gamma - \beta}\right) + p_P\left(\dfrac{p_B - p_P}{\gamma - \beta} - \dfrac{p_P - p_N}{\beta - \alpha}\right) + p_N\left(\dfrac{p_P - p_N}{\beta - \alpha} - \dfrac{p_N}{\alpha}\right)$ $\qquad$ (6)

$\qquad$ subject to: $\qquad\qquad\qquad\qquad\qquad\qquad p_B \leq p_P + p_N,$ $\qquad\qquad\qquad$ (7)

$$1 - (p_P - p_N)/(\beta - \alpha) \leq q_P, \text{ and} \qquad\qquad\qquad (8)$$

$$1 - (p_B - p_P)/(\gamma - \beta) + (p_P - p_N)/(\beta - \alpha) - p_N / \alpha \leq q_N. \qquad\qquad (9)$$

Clearly, the optimal strategy with unlimited resource availabilities (that is, when q_P and $q_N \geq 1/2$) is pure bundling. When the quality ratings are subadditive, this strategy is implemented with the prices p_B = $\gamma/2$, $p_P = \beta/2$, and $p_N = \alpha/2$. With superadditive quality ratings, the prices $p_B = \gamma/2$, $p_P = \beta/2$, and $p_N =$ $(\gamma - \beta)/2$ support pure bundling. These prices imply $T^* = T^{**} = T^{***} = 1/2$, and the seller's revenues are $\gamma/4$. Note that both these solutions also guarantee that the arbitrage constraint (7) is satisfied.

We now discuss the capacity constrained case in Proposition 3a (for subadditive ratings) and 3b (for superadditive ratings). We define the *propensity* to use a particular strategy as the area in the space of q_P and q_N values (see Figure 2) for which the strategy is applicable.

Proposition 3a. *When the relationship in the quality ratings is subadditive (i.e., when $\sigma < 1$),*

(i) *The full range of bundling strategies (PB, PC, MBPN, MBN$^-$, MBN$^+$, MBP$^-$, and MBP$^+$) can be optimal depending on the available capacities q_P and q_N (and hence ρ), and σ.*

(ii) *The propensity to use the PC and the MBPN strategies increases at the expense of the MBN$^-$ and MBP$^-$ strategies as the ratings subadditivity parameter σ decreases.*

(iii) *Higher subadditivity supports the MBPN strategy for greater discrepancy in the capacity levels.*

(iv) *MBN$^+$ is the optimal strategy if $0 < q_P < 1/2$, and $q_N \geq 1/2$. Similarly, MBP$^+$ is the optimal strategy if $0 < q_N < 1/2$, and $q_P \geq 1/2$.*

(v) *The pure bundling strategy, PB, is optimal at a single point $q_P = q_N = 1/2$.*

(vi) *Irrespective of the bundling strategy, no more than 1/2 of the total number of customers are served by the seller.*

Proposition 3b. *When the relationship in the quality ratings is superadditive* (*i.e., when* $\sigma \geq 1$),

(i)   *The MBPN and the PC strategies are never optimal. The PB strategy is optimal when* $\rho = 1$.

(ii)   *The optimal strategy does not depend on the quality relationship parameter,* σ.

According to Part (i) of Proposition 3a, all the different bundling strategies can be optimal when the quality ratings are subadditive. Indeed, these strategies arise actually in practice. For example, the American Bankers Association uses a full spectrum mixed bundling strategy, allowing advertisers to place advertisements in its magazines individually, or in a collection of its magazines (for a discount).[12] For the partial spectrum mixed bundling case, consider the popular video game console, Nintendo Wii. During the 2008 holiday season, many online retailers were not willing to sell the Wii (which was in short supply) by itself; they were offering it only as part of a bundle with Wii accessories, or the accessories by themselves. On the other hand, an established artist might use a pure components strategy by offering for sale only separate paintings (or sculptures). In this case, given the unique nature of these creations (and thus their scarcity), independent sale derives more value than does bundling. Finally, the Carnegie Museums of Pittsburgh is an umbrella organization for four different museums in Pittsburgh: Carnegie Museum of Art, Carnegie Museum of Natural History, The Andy Warhol Museum, and Carnegie Science Center. However, membership to individual museums is not sold, and *standard* benefits[13] of becoming a member of the Carnegie Museums of Pittsburgh include the membership of each of the four museums—an example of the use of pure bundling when the resource availability is unlimited. As another example with uncapacitated resources, cable companies only offer pure bundles of channels, and do not allow subscribers to order channels on an à la carte basis.

Figure 2 and the proof of the proposition (see Appendix) also facilitate computing the propensity to use the different strategies. With subadditivity, we find that the area of resource availability levels supporting the *PC* regime is $1/8(1 - \sigma)^2$, that supporting the *MBPN* regime amounts to $1/8(1 - \sigma^2)$, and

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[12] http://www.aba.com/bankmarketing/bm_advertising.htm
[13] http://www.carnegiemuseums.org/interior.php?pageID=45

that supporting partial mixed bundling regime amounts to $1/4(1 + \sigma)$. This supports and quantifies part

(ii) of Proposition 3a. Note from condition (a) in the proof of Proposition 3a (for subadditive ratings) that

when the available aggregate capacity is small (lower than $(1 - \sigma)/2$), offering the bundle is suboptimal

due to the extreme scarcity of the resources. When the aggregate capacity becomes larger than $(1 - \sigma)/2$,

and the capacity mismatch is relatively moderate ($\sigma \leq \rho \leq 1/\sigma$), it is optimal to choose full market

segmentation by offering all three different products. Condition (b) in the proof of Proposition 3a shows

that with increased subadditivity of the quality ratings (that is, smaller σ), full segmentation can be

supported by greater discrepancy between the resource capacities (that is, bigger $|q_N - q_P|$ values). On

the other hand, with reduced subadditivity (that is, larger σ values) full segmentation is optimal only

when the discrepancy between q_N and q_P is relatively small (Part (iii) of Proposition 3a).

When the availability of the normal resource is much greater than that of the premium resource

($\rho < \sigma$ when $\sigma < 1$ and $q_N > q_P$ when $\sigma \geq 1$), the seller offers a choice between the normal product and the

bundle. Conversely, when the availability of the premium resource is much greater than that of the

normal resource ($1/\sigma < \rho$ when $\sigma < 1$ and $q_P > q_N$ when $\sigma \geq 1$), the seller offers the premium product and

the bundle. With significant abundance of one resource relative to the other, it pays to utilize the entire

capacity of the more scarce resource as part of the bundle since buyers have a higher willingness to pay

for the bundle than for the separate components. Any remaining quantity of the more abundant resource,

not sold as part of the bundle, is offered separately to the customers. According to part (v) of Proposition

3a, pure bundling arises only when the capacity of each resource is large enough to obtain the

unconstrained model solution (equal to $1/2$).

Unlike the subadditive case, the *PC* and *MBPN* strategies are never optimal when the ratings are

superadditive, (Part (i) of Proposition 3b). The reason for this difference is that, with superadditivity the

bundle rating at least equals the sum of the ratings of its components, thus strongly incentivizing the seller

to offer the bundle to the fullest extent permitted by the resource availability. Consequently, the optimal

strategy is independent of the quality relationship parameter σ (as long as it is at least one), and the seller

implements the first best outcome of pure bundling only when both capacities are equal ($\rho = 1$). When a discrepancy in the capacities does exist, the seller utilizes the entire available capacity of the scarcer resource as a part of the bundle, and sells the leftover units of the more abundant resource as an independent component. With superadditivity and the consequent incentive to offer the largest feasible quantity of the bundle, the partial mixed bundling strategy is preferred to the *MBPN* strategy as it is closer to the pure bundling strategy.

Notice that the scarcity of the two resources, expressed in terms of $(1/2 - q_N)$ and $(1/2 - q_P)$, determines the regions in Figure 2a. These expressions measure the extent to which the individual capacities fall short of the unconstrained optimal value of 1/2, motivating also our definition of the capacity mismatch parameter ρ.

Having studied how the interaction between q_P and q_N, ρ and σ impacts the optimality of the different regimes, we now investigate the optimal pricing structure under the different strategies.

Proposition 4. *The optimal prices for the PC and MBPN strategies have the following structure:*

(i) *When the ratings are subadditive, the optimal prices are as follows:*

  a. *PC strategy:* $p_B = \gamma - \beta q_P - \alpha q_N$; $p_P = \beta(1 - q_P) - \alpha q_N$; *and* $p_N = \alpha(1 - q_P - q_N)$. (10)

  b. *MBPN strategy:*

$$p_B = \gamma/2 + (\beta - \alpha(1-\sigma)/(\sigma+1))(1/2 - q_P) + (2\alpha\sigma/(\sigma+1))(1/2 - q_N),$$
$$p_P = \beta/2 + (\beta - \alpha/(\sigma+1))(1/2 - q_P) + (\alpha\sigma/(\sigma+1))(1/2 - q_N), \qquad \text{and} \qquad (11)$$
$$p_N = \alpha/2 + (\alpha\sigma/(\sigma+1))(1 - q_P - q_N).$$

(ii) *The price arbitrage constraint is not binding in the PC and the MBPN regions when the ratings are subadditive, but is always binding when the ratings are superadditive.*

The proof of part (i) (see Appendix) follows by using the KKT conditions. The Appendix also provides the optimal product prices for the partial spectrum mixed bundling strategies. Note that with subadditive quality ratings the arbitrage constraint (7) is nonbinding for the *PC* and *MBPN* regions, and might possibly become binding in the MBN^+ and the MBP^+ regimes. If $1/2 < \sigma$ (that is, when the

subadditivity is moderate; note $\sigma < 1$), the arbitrage constraint might be violated when $q_P < 1/(2\sigma)$ under

MBN^+ or $q_N < 1/(2\sigma)$ under MBP^+. Since the arbitrage constraint is binding in this case, incorporating it

(that is, setting $p_P = p_B - p_N$ under MBN^+ or $p_N = p_B - p_P$ under MBP^+) still results in the desired

outcome for the seller. Specifically, no buyer chooses the premium (normal) product separately under

MBN^+ (MBP^+). With superadditive ratings, constraint (7) is always binding since the buyer's willingness

to pay for the bundle is always at least the sum of his willingness to pay for the individual components.

For the MBPN strategy (when by implication ratings are subadditive), Corollary 5 presents some

properties of the optimal solution.

Corollary 5.

For the MBPN strategy,

> (i) *The market share of the bundle decreases as the ratings subadditivity increases.*
>
> (ii) *Decreasing either capacity decreases the bundle quantity sold; the rate of this decrease is higher for higher subadditivity levels.*
>
> (iii) *Increasing either capacity decreases the prices of all three products.*
>
> (iv) *Increasing the quality rating of the*
>
> > a. *bundle increases the prices of all three products.*
> >
> > b. *premium product increases p_P but decreases p_N and p_B.*
> >
> > c. *normal product increases p_N but decreases p_P and p_B.*
>
> (v) *Increasing subadditivity of the ratings has an ambiguous effect on the product prices.*

Part (i) of Corollary 5 allows us to evaluate how the *MBPN* solution deviates from the "*first-best*"

outcome that we obtain in the unconstrained capacity model. Recall that in this case it is optimal to offer

only the bundle at a volume equal to 1/2. In contrast, for the constrained problem, the quantity of the

bundle sold is $1 - T^* = 1/2 - (1 - q_P - q_N)/(\sigma + 1)$. While the bundle is the only product sold in the

unconstrained case, the proportion of buyers that purchase the bundle is less than one in the constrained

case. The share of the bundle depends obviously upon the absolute degree of the scarcity (i. e.,

$1 - q_P - q_N$); and it also depends on the degree of subadditivity of the ratings, decreasing with increasing

subadditivity. From the expression of $1 - T^*$ we further note that each unit reduction in either resource

cuts back the bundle quantity at the rate of $1/(\sigma + 1) > 1/2$ (thus quantifying Part (ii) of Corollary 5) and

the individual component quantity at the complementary rate of $\sigma / (\sigma + 1) < 1/2$.

Interestingly, an increase in either the premium or the normal resource availability lowers the

optimal prices of all three products even though the seller offers a lower amount of the premium product

when q_N increases and of the normal product when q_P increases. However, the impact of higher quality

on the product prices is ambiguous.

Proposition 6 solves for the Lagrange multipliers of the resource constraints for the *MBPN*

strategy; we can derive the other shadow prices similarly. (We focus henceforth on just the case when the

ratings are subadditive because with superadditivity we do not get the full range of strategies.) This

analysis provides the foundation for our subsequent investigation into the relative marginal values of the

two resources.

Proposition 6. *The shadow prices of the resources for the MBPN solution are specified as follows:*

$$\lambda_P = \left(\beta - \alpha / (\sigma + 1)\right)\left(1 - 2q_P\right) + \left(\alpha\sigma / (\sigma + 1)\right)\left(1 - 2q_N\right); \quad \lambda_N = \left(2\alpha\sigma / (\sigma + 1)\right)\left(1 - q_P - q_N\right).$$

To understand the relationship between these shadow prices and the product prices, note that the

availability of one additional unit of a scarce resource results in both a direct effect of generating

additional revenues from the sale of this unit (partly separately and partly in the bundle), and an indirect

effect of depressing the prices that the seller can charge for the products. Specifically, since a marginal

increase in the premium resource is allocated to the bundle at the rate of $1/(\sigma + 1)$ and is sold separately at

the rate of $\sigma/(\sigma+1)$, an additional unit of the premium resource generates direct extra revenues equal to

$p_B/(\sigma + 1) + (p_P \sigma)/(\sigma +1)$. The extra unit depresses p_B at the rate of $\beta - \alpha(1 - \sigma)/(\sigma + 1)$ (Proposition 4).

Similarly, p_P is depressed at the rate of $\alpha\sigma/(\sigma + 1)$, and p_N at the rate of $\beta - \alpha/(\sigma + 1)$. Combining these

two effects yields the desired expression for λ_P, and similarly for λ_N. Interestingly, the shadow prices are

also dependent on the extent of subadditivity σ. Specifically, decreased subadditivity is associated with

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an increase in the valuation of both the premium and the normal resource.

Shadow prices measure the extra effort the seller might be willing to incur in order to obtain an additional unit of a scarce resource. In the case of television advertising, an increase in the available advertising time comes at the expense of programming time and thus can potentially decrease the ratings and hence the advertiser's profits (Wilbur, 2008). Similarly, expanding the retail space could have an adverse affect on the traffic congestion and parking space, and therefore decrease the number of shoppers. The shadow prices in Proposition 6 provide an upper bound on the reduction in quality ratings that a seller might be willing to tolerate in order to increase available resources by one unit.

We can use our results to evaluate how much more valuable an additional unit of the premium resource is vis-à-vis an additional unit of the normal resource, and how this added valuation changes as we move from one regime to another. (For the shadow prices for all the regions, see Banciu, Gal-Or and Mirchandani, 2010.) To answer this question, we use the difference $\lambda_P - \lambda_N$. We focus on the region $q_P < (1 - \sigma)/2$ since the premium resource has typically low availability. As well, in this region, increasing q_N shifts the optimal strategy according to Figure 2a from PC to $MBPN$ to MBN^- and finally to MBN^+. Figure 3 depicts the relative shadow prices of the two resources when considering such an increase in q_N. (The solid dots in this figure represent a shift in the strategy.)

<center>PLEASE INSERT FIGURE 3 ABOUT HERE</center>

The shadow price of the premium resource is higher than that of the normal resource since the seller can charge higher prices for the premium product. This higher price is proportional to the difference in the ratings, $(\beta - \alpha)$ under the PC and $MBPN$ regimes. Interestingly, the added market segmentation facilitated under $MBPN$ does not enhance the relative shadow price of the premium resource. The reason for this result is that an additional unit of either resource is allocated in the same proportion to the bundle, thus maintaining the relative desirability of the two resources. Under the MBN^+ regime, the difference $\lambda_P - \lambda_N$ is proportional to $(\gamma - \alpha)$ since this regime occurs under the extreme scarcity of the premium resource, and each additional unit of the premium resource is used only in the bundle, thus yielding the

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extra rating of γ rather than β. A similar allocation of an extra premium unit is optimal under MBN^- also. However, since there are no unused units of the normal resource under this regime, each additional unit of the bundle that is sold requires directing a normal product from being sold as an independent component. Consequently, λ_P under MBN^- is not as high as it is under MBN^+. Under MBN^-, the difference $\lambda_P - \lambda_N$ is an increasing function of q_N, or alternatively, since q_P is fixed for this analysis, an increasing function of ρ, until it reaches its maximum value when $q_N = 1/2$, and the MBN^+ region is reached. Note also that a larger q_P reduces the difference $\lambda_P - \lambda_N$ for all regimes. Hence, as the prime time becomes less scarce, its importance relative to the non-prime time resource declines.

In practice, we often see that companies are willing to pay a much higher price for increasing the capacity of the premium resource. Recently, in Pittsburgh,[14] it came to light that one of the largest billboard providers had allegedly given gifts to a city employee (who, as a consequence, was forced to resign for questionable ethics) to get him to sanction the construction of a downtown (premium location) digital billboard. (The billboard provider already had billboards in less populated Pittsburgh locations.) Similarly, magazine publishers incur a higher cost to create a fold-out inside cover or back page advertisement (premium space) versus simply adding an additional page in the magazine.

4. Horizontal Differentiation Among the Components

We now consider situations where due to horizontal differentiation *some* buyers derive additional benefits from particular products. For instance, advertisers may value both the number of viewers and their composition. Hence, even though prime-time attracts a greater number of viewers, some advertisers appreciate the profile of viewers during non-prime time (for example, retired professionals) due to the specific appeal of their products. Similarly, while all retailers may benefit from the higher shopper traffic at premium malls, some retailers may derive extra benefits by leasing a store in a normal mall if their products are more likely to appeal to the shoppers patronizing that mall.

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[14] http://www.thepittsburghchannel.com/news/15838339/detail.html

To incorporate this possibility, we assume that the population of buyers is divided into two segments. One segment, which we denote by type P, derives special idiosyncratic benefits from buying the premium variant of the product, and the other segment, denoted by type N, derives special benefits from buying the normal product. We model this possibility by assuming that for the former segment the quality ratings of the normal and premium variants are α and $\beta + \delta$, respectively, and for the latter segment they are $\alpha + \delta$ and β for some $\delta > 0$. Hence the added rating δ is a measure of the degree of horizontal differentiation between the two product variants. We assume that purchasing a bundle consisting of both product variants yields the quality ratings $\gamma + \delta$ for both market segments. Since the bundle contains each product variant, it incorporates the idiosyncratic benefit for each type of buyer.

Figure 4 depicts the quality ratings for each segment of buyers and distinguishes between two cases. Case 1 implies a relatively moderate degree of horizontal differentiation, so that the rating of the premium variant continues to be higher than that for the normal variant even for type N buyers (i.e., $\beta > \alpha + \delta$). In contrast, Case 2 applies when the degree of horizontal differentiation is high and the ranking of the two variants reverses for type N buyers. It is noteworthy that in the latter case not all three segments of buyers that are depicted in Figure 4 can survive. Specifically, if at the equilibrium $p_P > p_N$, type N buyers would not buy the premium variant as a stand-alone component since they are asked to pay a higher price for a product with a lower rating. In contrast, if at the equilibrium $p_N > p_P$, no type P buyer would purchase the normal product since it is more expensive and less valuable to this buyer type. In Case 1, however, all three segments for both types of buyers can survive in the MBPN region.

It is easy to show that the solution to the unconstrained optimization problem is still pure bundling even with the additional dimension of horizontal differentiation. In Proposition 7, we characterize the MBPN equilibrium when both resource capacity constraints are binding, assuming that

Case 1 of Figure 4 applies, namely that $\delta < \beta - \alpha$.[15]

Proposition 7. *When the number of type N and type P buyers is equal, $\sigma < 1$ and $\delta < \beta - \alpha$, the following is the characterization of the MBPN equilibrium:*

(i) *The MBPN region arises as an equilibrium if*

$$\sigma\left[\frac{(2\alpha+\delta)(\alpha\sigma+\delta)}{(\alpha+\delta)(2\alpha\sigma+\delta)}\right] \le \rho \le \frac{1}{\sigma}\left[\frac{(\alpha+\delta)(2\alpha\sigma+\delta)}{(2\alpha+\delta)(\alpha\sigma+\delta)}\right].$$

(ii) *The quantities Q_B, Q_P and Q_N of the bundle, premium and normal product variants sold at the equilibrium are*

$$Q_B = \frac{1}{2} - \frac{(\alpha+\delta)(2\alpha\sigma+\delta)(1-q_P-q_N)}{(\alpha(\sigma+1)+\delta)\left[(2\alpha+\delta)\sigma+\delta\right]}, \; Q_P = q_P - Q_B, \text{ and } Q_N = q_N - Q_B.$$

(iii) *A change in the extent of the horizontal product differentiation, as measured by δ, has an ambiguous effect on the quantities of products sold. Specifically,*

$$\frac{\partial Q_B}{\partial \delta} \text{ is } \begin{cases} > 0, \text{if } \delta \in \left[0, \alpha\sqrt{2\sigma}\right] \\ \le 0, \delta \in \left(\alpha\sqrt{2\sigma}, \beta-\alpha\right] \end{cases}; \quad \text{sgn}\left\{\frac{\partial Q_P}{\partial \delta}, \frac{\partial Q_N}{\partial r}\right\} = -\text{sgn}\left\{\frac{\partial Q_B}{\partial \delta}\right\}.$$

Part (i) of Proposition 7 implies that as the extent of the horizontal product differentiation increases, the size of the region supporting the MBPN regime shrinks. Relating this outcome to our earlier discussion, it appears that introducing an element of horizontal differentiation (while preserving the product ranking) has an effect that is equivalent to reducing the extent of ratings subadditivity (that is, increasing the quality relationship parameter, σ). Consequently, the ability of the seller to fully segment the population of the buyers decreases. Figure 5 illustrates this property with the arrows indicating how increasing horizontal differentiation changes the *MBPN* region.

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[15] The analysis for Case 2 (that is, when $\delta > \beta - \alpha$) leads to fairly long and complex expressions corresponding to the ones in Proposition 7 (and so we do not present them here). The complexity arises because of two reasons. First, a reversal in the product ranking occurs for Type N buyers resulting in asymmetric user preferences (and thus a shift from the vertically differentiated market structure). Second, as mentioned earlier, either Type P buyers do not buy the normal product or Type N buyers do not buy the premium product depending on the product prices.

According to parts (ii) and (iii), increased horizontal differentiation has an ambiguous effect on the optimal bundle quantity. With increasing horizontal differentiation, this quantity increases for small values of $\delta < \alpha\sqrt{2\sigma}$, and decreases for bigger values of $\delta > \alpha\sqrt{2\sigma}$. There are two counteracting effects that determine the optimal product mix as δ increases. Reducing subadditivity (Figure 5) implies that the seller should sell more of the bundle and less of the independent components, as dictated by the "first-best" outcome. However, as δ increases, the relative advantage of the bundle in comparison to the components declines, since the ratios $(\gamma + \delta) / (\alpha + \delta)$ and $(\gamma + \delta) / (\beta + \delta)$ become smaller as δ increases. It follows that the seller should increase the sale of the independent components. According to Proposition 7, the first effect of the reduced subadditivity dominates for relatively small values of δ, and the second effect of reduced advantage of the bundle dominates for relatively large values of δ. Note that when $\sigma > 1/2\left((\beta - \alpha)/\alpha\right)^2$, the quantity of the bundle sold unambiguously increases as δ increases since $\alpha\sqrt{2\sigma} > \beta - \alpha$ in this case. Hence, if the intrinsic relationship between the quality ratings is only modestly subadditive, adding a dimension of horizontal differentiation unambiguously yields a solution that is closer to the "first best" outcome of pure bundling.

It is noteworthy that our assumption that the bundle always fully incorporates the idiosyncratic benefits derived by buyers may not always be valid. A luxury fashion house such as Louis Vuitton may actually lose some of its cachet as an exclusive retailer by leasing a store in a normal mall. On the other hand, a premium mall location may have a detrimental effect on Claire's image of a store that sells affordable costume jewelry, and thus on its appeal to its core base of tween and teenage girls. In these two examples, the bundle rating may increase by less than δ. Moreover, it is even possible that such retailers value leasing a store in their preferred mall to leasing stores in both malls (because $\beta + \delta > \gamma + \delta_1$ for Louis Vuitton and $\alpha + \delta > \gamma + \delta_2$ for Claire's, where $\gamma + \delta_1$ and $\gamma + \delta_2$ are the respective modified ratings of the bundle). With such significant levels of horizontal differentiation, the segmentation of

buyers depicted in Figure 4 becomes much more difficult.[16]

5. Extension: General Density Functions

We now check the sensitivity of our results to changes in the density function of the efficiency random variable. To do so, we use the family of (standard) Beta distributions because it has the domain [0, 1] which equals our assumed efficiency range, and changing the parameter values generates the different shapes that are interesting from our perspective. The Beta distribution has two shape parameters, which we denote by a and b. Figure 5 gives the parametric settings and the four different shapes that we will investigate. Given the complexity of deriving the analytic solution for these more general density functions, we complement our analytical results with numerical computations. We assume that $\alpha = 1$, $\beta = 2$, and $\gamma = 2.5$ in order to focus on a subadditive relationship in the quality ratings. Since $\alpha = 1$ and $\beta = 2$, γ must lie in the open interval (2, 3) and so a value of 2.5 for γ denotes "medium" incentive to bundle, thus not favoring either a *PB* or a *PC* strategy.

PLEASE INSERT FIGURE 6 ABOUT HERE

We refer to the buyers having the efficiency distribution in Figure 6 (a) as *parsimonious* buyers because a large majority of them have a low willingness to pay. Similarly, we refer to buyers in Figures 6 (b), 6 (c), and 6 (d) as *centric* buyers, *uniform* buyers and *high-spenders* respectively (we have shown the uniform distribution in this figure for consistency with our later figures).

Figure 7 presents the seller's optimal strategies (determined numerically) for each of the four different buyer types as the availability of the two resources changes. Even though there are differences across the different distribution types that reflect the distributions' unique characteristics, the general structure of the optimal strategies is similar.

Comparing the parsimonious buyers and the high-spenders cases (Figures 7 (a) and 7 (d)) we

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[16] An alternate way of modeling horizontal differentiation is to use a multiplicative model for incorporating horizontal differentiation. In this case, for $\mu > 1$, the quality ratings for a type *P* (type *N)* buyer are α, $\mu\beta$, and $\mu\gamma$ ($\mu\alpha$, β, and $\mu\gamma$) for the normal, premium and bundle products respectively.

observe that the *PC* region is smaller for parsimonious buyers. The reason for this difference is that parsimonious buyers are concentrated near the low end of the efficiency scale. In order to extract greater revenue from them, the seller offers full spectrum mixed bundling even when both resource availabilities are low (and the relative availabilities are about the same). For the high-spenders case, the seller uses the *PC* strategy for a greater range of resource availabilities because quality ratings are subadditive.

As we mentioned earlier for the uniform buyers case, unconstrained optimization corresponds to both q_N and q_P values being at least a half. For the parsimonious buyers case, we can use Proposition 1 to show that the unconstrained region begins at $q_N = q_P = 4/9$. Figure 7 (a) reflects this observation. For the high-spenders case, again using Proposition 1, we can show that the unconstrained region begins at $q_N = q_P = 2/3$. Just like for the uniform buyer case, these values of 4/9 and 2/3 do not seem to depend on the value of γ. Thus, the pure bundle is not offered for the high-spenders case when the sum of the resource availabilities is at most one, as we have assumed in this paper. Schmalensee (1984) has previously observed that mixed bundling reduces the heterogeneity in the customers, and therefore allows better price discrimination. A natural measure of heterogeneity is variance, and the distributions for both parsimonious buyers and high-spenders have equal variances. Yet, for parsimonious advertisers, pure bundling is the optimal strategy for a larger region defined by q_N and q_P, and for high-spenders, mixed bundling is the optimal strategy for a larger region. This comparison of Figures 7 (a) and 7 (d) thus demonstrates that the *skewness* of the efficiency distribution, besides its heterogeneity, seems to affect the benefits of mixed bundling.

PLEASE INSERT FIGURE 7 ABOUT HERE

6. Conclusions

In this paper, we have examined bundling strategies when products are vertically differentiated and the underlying resource capacities are limited. While this research is motivated by an application from the retail industry, the institutional characteristics of television advertising and several other situations are similar. We analyze both the situation when the quality ratings of the products are

subaddivie and when they are superadditive. Our results show that the relative availabilities of the resources and the subadditivity of the ratings strongly influence the seller's optimal strategy of implementing full spectrum mixed bundling (offering the bundle and each of the components), or partial spectrum mixed bundling (offering the bundle with one of the components), or not using bundling at all. When the ratings are superadditive, the full spectrum mixed bundling and the pure components strategies are never optimal. These results differ from those in the extant bundling literature which has not considered either vertical differentiation or limited capacities. We also investigate how introducing horizontal differentiation affects the optimal strategies. In particular, we find that increased horizontal differentiation has an effect similar to reduced subadditivity, and thus the propensity to use the first-best solution of pure bundling increases with increasing horizontal differentiation. Finally, we examine the robustness of our analytical conclusions to more general distributions of buyers using numerical testing, and find that the propensity for various bundling strategies is influenced not only by the heterogeneity of customer preferences, but also by the skewness of this distribution.

Our research points towards several promising research directions. First, we have assumed a monopolistic environment with only one seller. Introducing competition, where buyers desiring to purchase, say, the premium product have a choice of multiple sellers, adds interesting nuances. There is current literature (Economides, 1993; Kopalle, Krishna and Assuncao, 1999; Matutes and Regibeau, 1992) that has studied bundling in the presence of competition. However, this research assumes complementary products, not a vertically differentiated market; and this literature does not account for scarce resources. The previous work does not obtain the full range of bundling strategies that we derive. It would be interesting to study how the range of strategies manifests itself when the resources are limited in a competitive sellers' market. Second, incorporating additional objectives of the buyer into the bundling framework also promises to be interesting and challenging. For example, a retailer might need to include factors such as store clustering, store dispersal and market saturation (see, for example, Vandell and Carter (1993)) in the decision process. Similarly, advertisers' objectives based on recent work in targeted advertising (Chen and Iyer, 2002; Gal-Or and Gal-Or, 2005; Gal-Or et al., 2006; Iyer, Soberman

and Villas-Boas, 2005), or combative advertising (Chen et al., 2009) might be relevant in developing the optimal bundling strategy. Third, we have assumed that the resource capacities are limited and that their marginal costs are zero (or, equivalently, that the resource availabilities are limited and the resource costs are sunk). It might be worth investigating how the results change if this marginal cost assumption does not hold. Fourth, it might be useful to investigate the optimal bundling strategies in the presence of multiple (more than two) resource classes (for example, urban, suburban and rural malls, or in internet advertising, the number of clicks needed from the home page to reach the advertisement). Fifth, our model is deterministic along the buyers' willingness to pay; introducing stochastic elements with respect to this dimension (Ansari, Siddarth and Weinberg, 1996; Venkatesh and Mahajan, 1993) might also be worthwhile. Finally, in this paper, each component requires only one resource. While this assumption is appropriate for the applications we have cited, it may be useful to also investigate situations in which the components each require an idiosyncratic resource and a second resource common to the components and perhaps the bundle.[17] These and other nuances associated with bundling, as well as the challenge in modeling and analyzing bundling situations and its inter-disciplinary appeal, will in all likelihood guarantee that bundling will continue to be a fertile research area.

Appendix

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[17] For example, Samsung Australia's current promotions include one in which a 26 inch, older technology, LCD television is bundled with a larger, newer technology, LED television (http://www.samsung.com/au/tv/). This market is vertically differentiated, and the availabilities of the two television types are limited. In addition, though, a common resource such as the seller's total distribution capacity could influence the optimal bundling strategy.

Proof of Proposition 1 and Corollary 2: The necessary first order conditions on the unconstrained total revenue function (1) yield

$$\partial \pi / \partial p_B = 0 \Leftrightarrow 1 - F\left(T^*\right) - T^* f\left(T^*\right) = 0,$$

$$\partial \pi / \partial p_P = 0 \Leftrightarrow F\left(T^*\right) - F\left(T^{**}\right) + T^* f\left(T^*\right) - T^{**} f\left(T^{**}\right) = 0, \text{ and}$$

$$\partial \pi / \partial p_N = 0 \Leftrightarrow F\left(T^{**}\right) - F\left(T^{***}\right) + T^{**} f\left(T^{**}\right) - T^{***} f\left(T^{***}\right) = 0.$$

We can write the optimality conditions listed above as:

$$[1 - F(T^*)][1 - T^*h(T^*)] = [1 - F(T^{**})][1 - T^{**}h(T^{**})] = [1 - F(T^{***})][1 - T^{***}h(T^{***})] = 0.$$

We ignore the trivial solution $T^* = T^{**} = T^{***} = 1$ to this system of equations. Next, note that the domain of t is [0, 1]; moreover, $[1 - th(t)]$ equals 1 when $t = 0$, and since $h(1) > 1$, $[1 - th(t)]$ is negative when $t = 1$. Hence, the intermediate value theorem implies that there is at least one fixed point T^* that satisfies $T^* = 1/h(T^*)$ for the first equation. Because $th(t)$ is monotonically increasing, this fixed point is unique and equals the fixed points T^{**} and T^{***} corresponding to the second and the third equations. Therefore, $p_B = \gamma T^*$ and the revenue function equals $\gamma T^* [1 - F(T^*)]$. Lariviere (2006) shows that the function $\gamma t[1 - F(t)]$ is unimodal when $th(t)$ is monotonically increasing. Hence, $T^* = 1/h(T^*) = T^{**} = T^{***}$ is a global maximum, the optimal strategy is pure bundling, the optimal bundle price is γT^*, and the optimal revenue is $\pi = \gamma T^*[1 - F(T^*)] = \gamma[1 - F(T^*)] / h(T^*)$.

Using Markov's inequality, we can find an upper bound on the optimal revenue as follows:

$$\Pr\left(t > a\right) \le \mathbf{E}\left[T\right] / a \Leftrightarrow \gamma T^* \Pr\left(t > T^*\right) \le \gamma T^* \mathbf{E}\left[T\right] / T^* \Leftrightarrow \pi \le \gamma \mathbf{E}\left[T\right].$$

Remark: Since t is nonnegative, $th(t)$ is monotonically increasing in t if the hazard rate function $h(t)$ is monotonically increasing in t. (The function $th(t)$ could be monotonically increasing even when $h(t)$ is decreasing.) The function $h(t)$ is monotonically increasing whenever the density function $f(t)$ exhibits Log-concavity (see, for example, Bagnoli and Bergstrom (2005)). Bagnoli and Bergstrom list a large number of commonly used distributions such as the Uniform, Normal, Beta (with both its shape parameters at least 1), Power, Logistic, Exponential, Laplace, Chi-squared, Chi, Gamma, and Weibull that along with their truncations (both left and right) and linear transformations satisfy the Log-concavity

property. We find that $h(1) > 1$ for the Beta distribution which has support [0, 1], and also for the truncations between 0 and 1 of all other distributions in this list. This observation demonstrates that pure bundling is the optimal strategy for a large number of commonly used distributions in the unconstrained capacity case.

Proof of Proposition 3a ($\sigma < 1$): To establish the observations in Proposition 3a, we first note that the different strategies are optimal as follows:

(a)   The pure component strategy, PC, is optimal if $0 < q_N + q_P < (1-\sigma)/2$.

(b)   The full spectrum mixed bundling strategy, MBPN, is optimal if $q_N + q_P \geq (1-\sigma)/2$ and

$\sigma \leq \rho \leq 1/\sigma$.

(c)   The partial spectrum mixed bundling strategies, MBN⁻ and MBN⁺, are optimal if $0 < q_P <$

1/2, and (c1) $\rho < \sigma$ and $q_N < 1/2$ for MBN⁻, and (c2) $q_N \geq 1/2$ for MBN⁺.

(d)   The partial spectrum mixed bundling strategies, MBP⁻ and MBP⁺, are optimal if $0 < q_N <$

1/2 and (d1) $\rho > 1/\sigma$ and $q_P < 1/2$ for MBP⁻, and (d2) $q_P \geq 1/2$ for MBP⁺.

(e)   The pure bundling strategy, PB, is optimal at a single point $q_P = q_N = 1/2$.

To establish this, observe that the *ROMB_U* model is a quadratic optimization program with linear constraints; therefore the first-order KKT conditions are both necessary and sufficient. The proof is straightforward once the first order conditions are expressed under all capacity scenarios: no binding capacity constraints; one binding capacity constraint (two cases), and both capacity constraints are binding. For brevity, we present the optimal thresholds only for the last case:

$$T^* = 1/2 + \left(1/(\sigma+1)\right)\left(1 - q_P - q_N\right), T^{**} = 1 - q_P, \text{ and } T^{***} = 1/2 + \left(\sigma/(\sigma+1)\right)\left(1 - q_P - q_N\right).$$

The thresholds satisfy $0 \leq T^{***} \leq T^{**} \leq T^* \leq 1$. Therefore, for example, the last inequality is equivalent to $T^* \leq 1 \Leftrightarrow \left(1/(\sigma+1)\right)\left(1 - q_P - q_N\right) \leq 1/2 \Leftrightarrow q_P + q_N \geq (1-\sigma)/2$.

The bundle is not offered when equality holds, therefore condition (a) above follows. Conditions (b), (c) and (d) are derived similarly from the remaining inequalities involving the thresholds, and

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observing that either MBN^+ or MBP^+ strategies imply only one capacity constraint binding. Finally, using Proposition 1 with $F(x) = x$ and $f(x) = 1$, we obtain $T^* = T^{**} = T^{***} = 1/2$ and the substitution into both capacity constraints yields condition (e).

Part (i) of Proposition 3a follows immediately because $\sigma < 1$, Part (ii) and (iii) follow from (a) and (b) above, Part (iv) from (d), Part (v) is the same as (e) above, and Part (vi) follows from the T^{***} values.

Proof of Proposition 3b ($\sigma \geq 1$): We first demonstrate that the PC and the $MBPN$ regions can never arise as an optimal strategy when the quality ratings are superadditive. Suppose that the solution yields the PC strategy at equilibrium. The invariant conditions (5) and the full utilization of the resources yield that the prices of the three products are given by (10). However, note that $p_B \geq p_P + p_N$ in (10) if $\gamma \geq \alpha + \beta$. Hence, the arbitrage constraint is binding and $p_B = p_P + p_N$. Including this equality in the optimization implies that the bundle will be chosen by all buyers, leading to a contradiction. Similarly, if $MBPN$ is an equilibrium, equation (11) supports it as an outcome. However, in this case as well, $p_B \geq p_P + p_N$ when the ratings are superadditive. Incorporating the binding constraint $p_B = p_P + p_N$ yields, once again, that segmentation cannot be supported since any consumer who chooses a component (either the normal or premium) would be better off deviating to choosing the bundle. Hence, with superadditive ratings the arbitrage constraint is binding for any strategy that supports segmentation. Solving the optimization problem when $p_B = p_P + p_N$ yields that pure bundling should be implemented to the fullest extent supported by the availability of the resources. When $q_P = q_N$, it can be fully supported. With asymmetry in the availability of the resources the MBN and MBP regimes are the closest that the seller can reach to pure bundling. Thus,

(a) *The pure bundling strategy, PB, is optimal whenever $q_P = q_N$.*

(b) *The partial spectrum mixed bundling strategies, MBN^- and MBN^+, are optimal if $0 < q_P < 1/2$, and (b1) $q_P < q_N < 1/2$ for MBN^- and (b2) $q_N \geq 1/2$ for MBN^+.*

(c) *The partial spectrum mixed bundling strategies, MBP^- and MBP^+, are optimal if $0 < q_N <$*

1/2, and (c1) $q_N < q_P < 1/2$ for MBP^- and (c2) $q_P \geq 1/2$ for MBP^+.

Proof of Proposition 4 and Corollary 5: When the relationship between the ratings is subadditive, we use the fact that the first order conditions are both necessary and sufficient. Additionally, the invariant boundary conditions (5) establish the relationships between thresholds and prices. For brevity, under the case of both capacity constraints binding, we derive the optimal price for the non-prime time product:

$$\left.\begin{array}{l} \alpha T^{***} - p_N = 0 \Leftrightarrow p_N = \alpha T^{***} \\ T^{***} = 1/2 + \left(\sigma/(\sigma+1)\right)\left(1 - q_P - q_N\right) \end{array}\right\} \Rightarrow p_N = \alpha/2 + \left(\sigma/\left(\alpha(\sigma+1)\right)\right)\left(1 - q_P - q_N\right).$$

Similarly we obtain the remaining prices as

$$p_P = \beta/2 + \left(\beta - \alpha/(\sigma+1)\right)\left(1/2 - q_P\right) + \left(\alpha\sigma/(\sigma+1)\right)\left(1/2 - q_N\right), \text{ and}$$
$$p_B = \gamma/2 + \left(\beta - \alpha(1-\sigma)/(\sigma+1)\right)\left(1/2 - q_P\right) + \left(2\alpha\sigma/(\sigma+1)\right)\left(1/2 - q_N\right).$$

The optimal prices for *PC* case, as well as for the other strategies given below, follow similarly.

STRATEGY	OPTIMAL PRICES
MBN^-	$p_B = \gamma(1 - q_P) - \alpha(q_N - q_P); \ p_P = \beta(1 - q_P) - \alpha(q_N - q_P); \ p_N = \alpha(1 - q_N)$ Superadditive Case: $p_P = (\gamma - \alpha)(1 - q_P)$
MBN^+	$p_B = \gamma/2 + (\gamma - \alpha)(1/2 - q_P); \ p_P = \max\left\{\alpha/2 + (\beta - \alpha)(1 - q_P), (\gamma - \alpha)(1 - q_P)\right\}; \ p_N = \alpha/2$ Superadditive Case: $p_P = (\gamma - \alpha)(1 - q_P)$
MBP^-	$p_B = \gamma(1 - q_N) - \beta(q_P - q_N); \ p_P = \beta(1 - q_P); \ p_N = \alpha(1 - q_P)$ Superadditive Case: $p_N = (\gamma - \beta)(1 - q_N)$
MBP^+	$p_B = \gamma/2 + (\gamma - \beta)(1/2 - q_N); \ p_P = \beta/2; \ p_N = \max\left(\alpha/2, (\gamma - \beta)(1 - q_N)\right)$ Superadditive Case: $p_N = (\gamma - \beta)(1 - q_N)$

Recall that the *PC* and the *MBPN* cases are not applicable when ratings are superadditive, and the product prices for the other cases equal the corresponding prices for the subadditive case unless they are mentioned in the table above. Also, when the ratings are superadditive only the *MBN* and *MBP* regimes can survive if $q_P \neq q_N$. The derivation of the prices in this case is similar to when the ratings are subadditive, with the exception that the arbitrage constraint is always binding. Hence, the price of the variant of the product that is not offered in each regime is determined by the equation $p_B = p_P + p_N$.

The proof of Corollary 5 follows immediately from the *MBPN* threshold and price values.

Proof of Proposition 6: The proof relies on the first order conditions and solving for the Lagrange multipliers.

Proof of Proposition 7: Using the notation of Figure 4, the thresholds can be expressed as follows:

$$T_N^* = \frac{p_B - p_P}{\gamma + \delta - \beta}, T_N^{**} = \frac{p_P - p_N}{\beta - \alpha - \delta} \text{ and } T_N^{***} = \frac{p_N}{\alpha + \delta}; \text{ and } T_P^* = \frac{p_B - p_P}{\gamma - \beta}, T_P^{**} = \frac{p_P - p_N}{\beta + \delta - \alpha} \text{ and } T_P^{***} = \frac{p_N}{\alpha}.$$

Substituting the above into the constrained maximization problem assuming the *MBPN* regime and an equal number of type *N* and type *P* buyers yields

$$\max \pi = p_B\left(1 - \frac{(p_B - p_P)(\gamma - \beta + \alpha/2)}{(\gamma + \delta - \beta)(\gamma - \beta)}\right) + p_P\left(\frac{(p_B - p_P)(\gamma - \beta + \alpha/2)}{(\gamma + \delta - \beta)(\gamma - \beta)} - \frac{(p_P - p_N)(\beta - \alpha)}{(\beta + \delta - \alpha)(\beta - \alpha - \delta)}\right)$$
$$+ p_N\left(\frac{(p_P - p_N)(\beta - \alpha)}{(\beta + \delta - \alpha)(\beta - \alpha - \delta)} - \frac{p_N(\alpha + \delta/2)}{\alpha(\alpha + \delta)}\right)$$

subject to: $\qquad p_B - (p_P + p_N) \le 0, \qquad\qquad 1 - \frac{(p_P - p_N)(\beta - \alpha)}{(\beta + \delta - \alpha)(\beta - \alpha - \delta)} \le q_P,$ and

$$1 - \frac{(p_B - p_P)(\gamma - \beta + \alpha/2)}{(\gamma + \delta - \beta)(\gamma - \beta)} + \frac{(p_P - p_N)(\beta - \alpha)}{(\beta + \delta - \alpha)(\beta - \alpha - \delta)} - \frac{p_N(\alpha + \delta/2)}{\alpha(\alpha + \delta)} \le q_N.$$

With subadditive ratings, the first order conditions are necessary and sufficient. Solving the above optimization problem yields prices and the threshold conditions as follows:

$$p_B = \gamma - \frac{\alpha^2}{2\alpha + \delta} - \frac{\delta^2(1 - q_P)}{\beta - \alpha} - (\beta - \alpha)q_P - \frac{(\gamma - \beta)^2}{2\gamma - 2\beta + \delta} + \frac{4\alpha(\alpha + \delta)(\gamma - \beta)(\gamma - \beta + \delta)(1 - q_P - q_N)}{(\gamma - \beta + \alpha + \delta)[(2\alpha + \delta)(\gamma - \beta) + \delta\alpha]},$$

$$p_P = \beta - \frac{\alpha^2}{2\alpha + \delta} - \frac{\delta^2(1 - q_P)}{\beta - \alpha} - (\beta - \alpha)q_P + \frac{2\alpha(\alpha + \delta)(\gamma - \beta)(\gamma - \beta + \delta)(1 - q_P - q_N)}{(\gamma - \beta + \alpha + \delta)[(2\alpha + \delta)(\gamma - \beta) + \delta\alpha]},$$

$$p_N = \frac{\alpha(\alpha + \delta)}{2\alpha + \delta} + \frac{2\alpha(\alpha + \delta)(\gamma - \beta)(\gamma - \beta + \delta)(1 - q_P - q_N)}{(\gamma - \beta + \alpha + \delta)[(2\alpha + \delta)(\gamma - \beta) + \delta\alpha]},$$

$$T^* = \frac{T_P^* + T_N^*}{2} = \frac{1}{2}\left(\frac{p_B - p_P}{\gamma - \beta} + \frac{p_B - p_P}{\gamma + \delta - \beta}\right) = \frac{1}{2} + \frac{\alpha(\alpha + \delta)(2\gamma - 2\beta + \delta)(1 - q_P - q_N)}{(\gamma - \beta + \alpha + \delta)[(2\alpha + \delta)(\gamma - \beta) + \delta\alpha]},$$

$$T^{**} = \frac{T_P^{**} + T_N^{**}}{2} = \frac{1}{2}\left(\frac{p_P - p_N}{\beta + \delta - \alpha} + \frac{p_P - p_N}{\beta - \alpha - \delta}\right) = 1 - q_P, \text{ and}$$

$$T^{***} = \frac{T_P^{***} + T_N^{***}}{2} = \frac{1}{2}\left(\frac{p_N}{\alpha} + \frac{p_N}{\alpha + \delta}\right) = \frac{1}{2} + \left(\frac{\gamma - \beta}{\gamma - \beta + \alpha + \delta}\right)\left[1 + \frac{\delta(\alpha + \delta)}{(2\alpha + \delta)(\gamma - \beta) + \delta\alpha}\right](1 - q_P - q_N).$$

Note that the prices derived above guarantee the arbitrage constraint $p_B - (p_P + p_N) \le 0$ when

$\sigma < 1$. To ensure the *MBPN* regime, we need $T^* > T^{**} > T^{***}$. Imposing the conditions yields part (i) of

the proposition. The second part follows since $Q_B = 1 - T^*$, $Q_P = T^* - T^{**}$, and $Q_N = T^{**} - T^{***}$.

Differentiating these expressions with respect to δ yields part (iii).

<div align="center">

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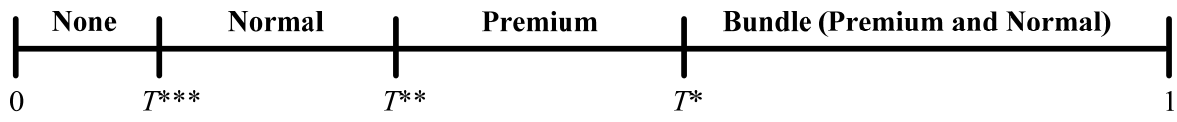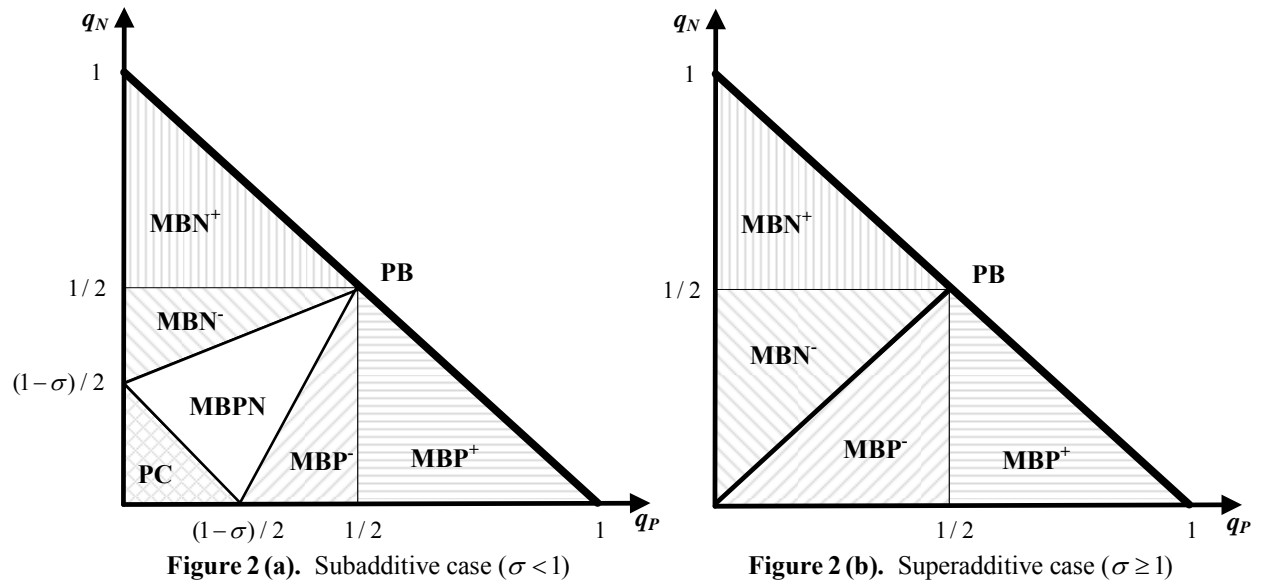
Figure 1. Market segmentation.

Figure 2 (a). Subadditive case ($\sigma < 1$)   **Figure 2 (b).** Superadditive case ($\sigma \geq 1$)

Figure 2. Representation of the optimal strategies for both subadditive and superadditive cases.
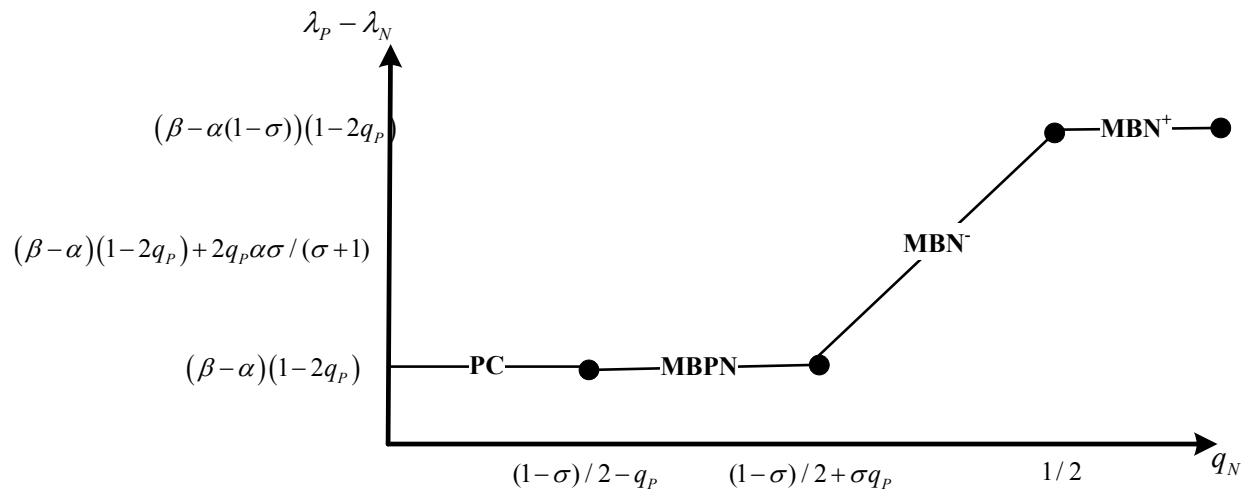
Figure 3. Relative shadow prices of the resources for any $q_P < (1 - \sigma)/2$.
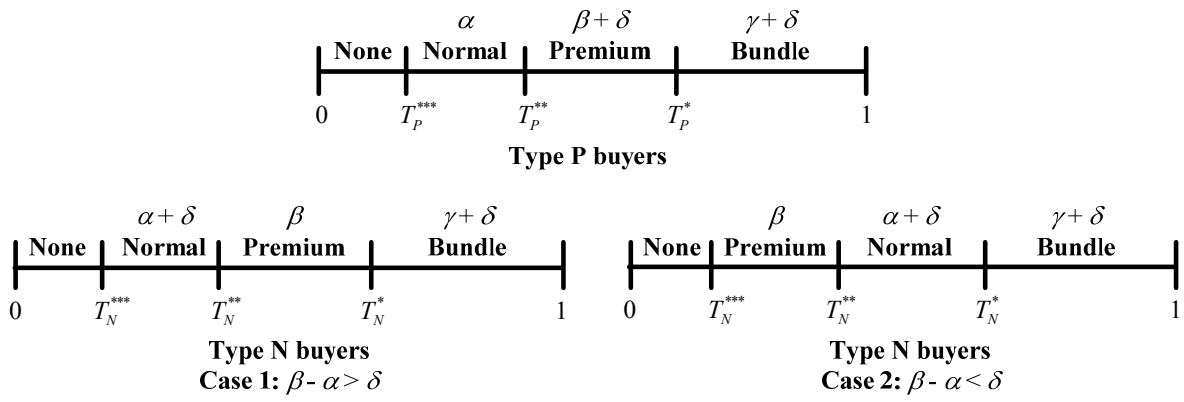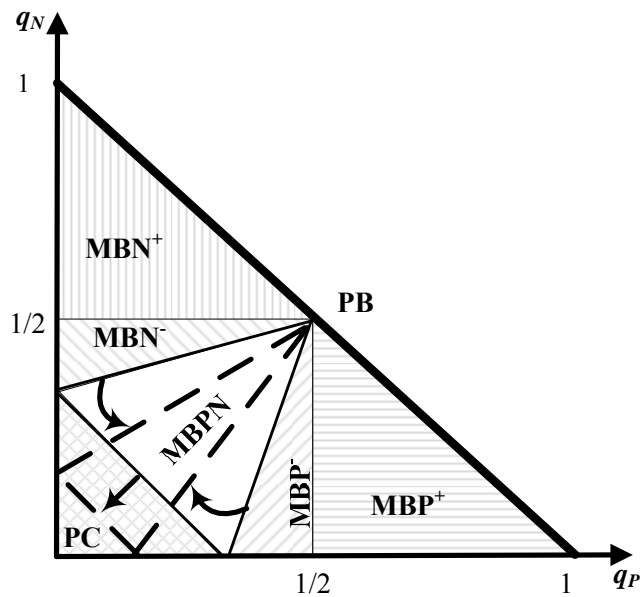
Figure 4. Segmentation of buyers.

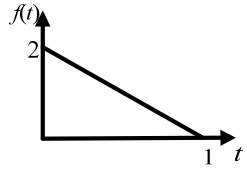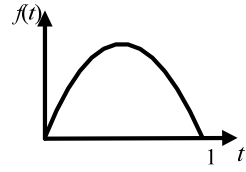**Figure 5.** The effect of increasing horizontal differentiation on the extent of segmentation.

Figure 6 (a): *a*=1, *b*=2   **Figure 6 (b):** *a*=2, *b*=2   **Figure 6 (c)**: *a*=1, *b*=1   **Figure 6 (d):** *a*=2,*b*=1
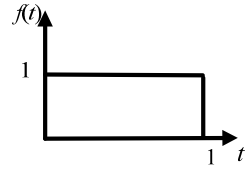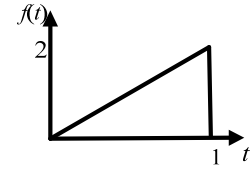
Figure 6. Density functions.
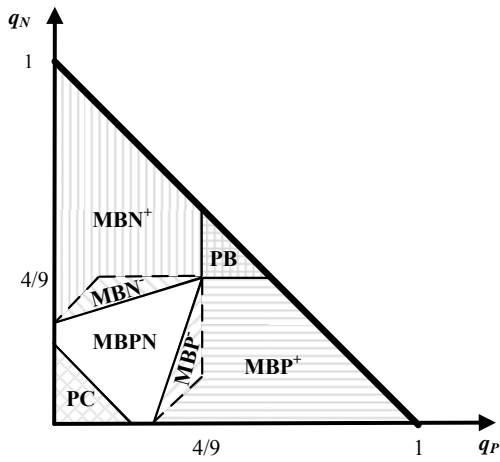
Figure 7 (a): Parsimonious buyers
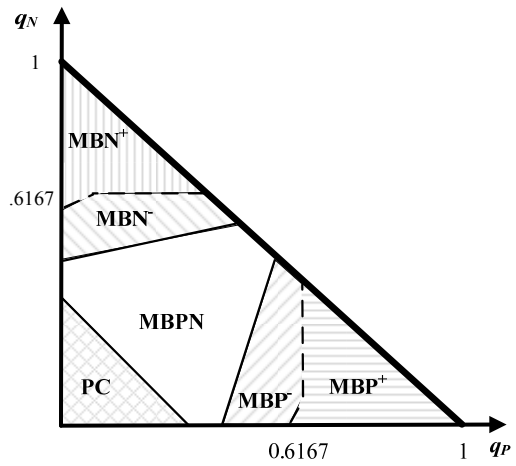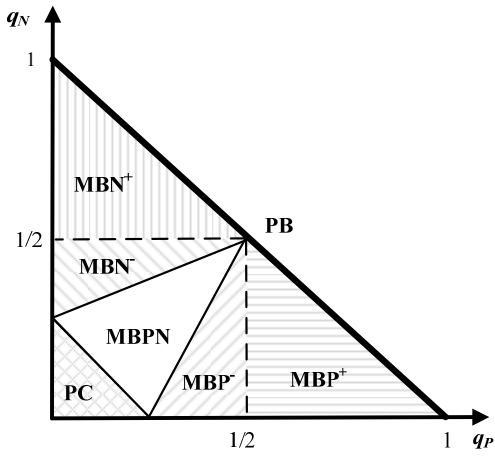

Figure 7 (b): Centric buyers
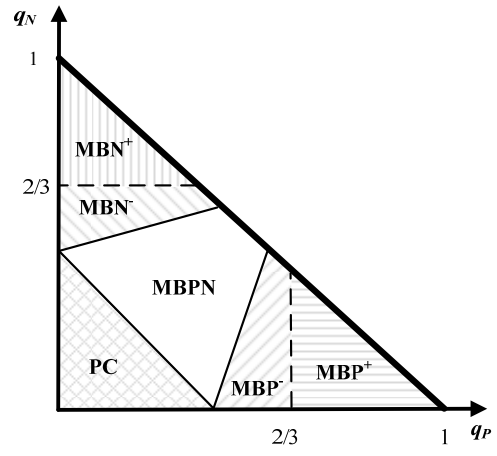

Figure 7 (c): Uniform buyers

Figure 7 (d): High-spenders

Figure 7. Strategies for the different buyer types.